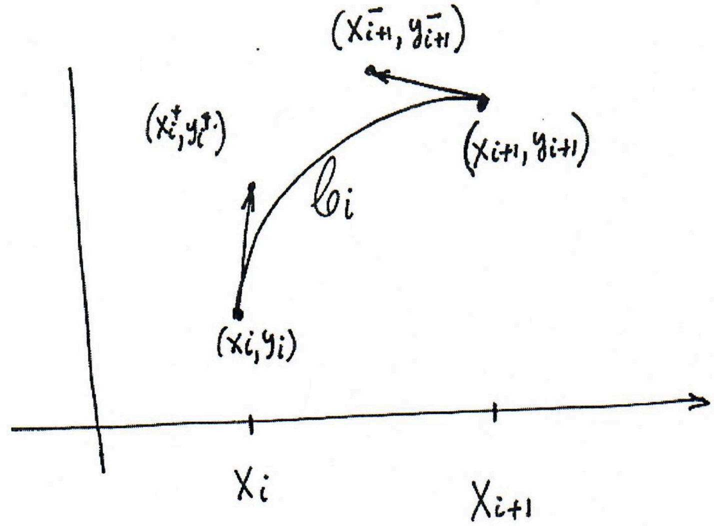
Bezier curves



Interpolation points are : (x_i, y_i), (x_{i+1}, y_{i+1})

Guidepoints are: (x_i^+, y_i^+), (x_{i+1}^-, y_{i+1}^-)

Hermite Polynomials for the parametric equations:

$$\ell_i : \begin{cases} x_i(t) = a_0^{(i)} + a_1^{(i)} t + a_2^{(i)} t^2 + a_3^{(i)} t^3 & (1) \\ y_i(t) = b_0^{(i)} + b_1^{(i)} t + b_2^{(i)} t^2 + b_3^{(i)} t^3 & (2) \end{cases} \qquad t \in [0,1]$$

They satisfy the following BCs :

$x_i(0) = x_i$	$y_i(0) = y_i$	(3)
$x_i(1) = x_{i+1}$	$y_i(1) = y_{i+1}$	(4)
$x_i'(0) = x_i^+ - x_i$	$y_i'(0) = y_i^+ - y_i$	(5)
$x_i'(1) = x_{i+1} - x_{i+1}^-$	$y_i'(1) = y_{i+1} - y_{i+1}^-$	(6)

by multiplying by a constant all the derivatives at the ends,(*) the slopes of the tangents to the curve $\mathcal{C}_i$ at (x_i, y_i) and (x_{i+1}, y_{i+1}) do not change. For Bezier curves, that constant is "3." It means

$$x_i'(0) = 3\,(x_i^+ - x_i) \quad , \quad y_i'(0) = (y_i^+ - y_i)\,3 \quad (5.2)$$

$$x_i'(1) = 3\,(x_{i+1} - x_{i+1}^-) \quad , \quad y_i'(1) = (y_{i+1} - y_{i+1}^-)\,3 \quad (6.2)$$

Using the 8 conds. the eight coefficients are determined.

$$x_i(t) = x_i + 3(x_i^+ - x_i)t + 3(x_i + x_{i+1}^- - 2x_i^+)\,t^2 + \left(x_{i+1} - x_i + 3(x_i^+ - x_{i+1}^-)\right)t^3 \quad (7)$$

$$y_i(t) = y_i + 3(y_i^+ - y_i)t + 3(y_i + y_{i+1}^- - 2y_i^+)\,t^2 + \left(y_{i+1} - y_i + 3(y_i^+ - y_{i+1}^-)\right)t^3 \quad (8)$$

(*) what it changes is the magnitude of the tangent vector to $\mathcal{C}_i$ at each end.

Determination of the coefficients: a's and b's

$$\begin{cases} x_i'(t) = a_1^{(i)} + 2a_2^{(i)}t + 3a_3^{(i)}t^2 \\ y_i'(t) = b_1^{(i)} + 2b_2^{(i)}t + 3b_3^{(i)}t^2. \end{cases}$$

Then

$$\left.\begin{array}{l} x_i'(0) = 3(x_i^{+} - x_i) \\ \text{and } x_i'(0) = a_1^{(i)} \end{array}\right\} \Rightarrow \boxed{a_1^{(i)} = 3\,(x_i^{+} - x_i)}$$

Similarly, $\boxed{b_1^{(i)} = 3\,(y_i^{+} - y_i)}$

Also,

$$3(x_{i+1} - x_{i+1}^{-}) = x_i'(1) = a_1^{(i)} + 2a_2^{(i)} + 3a_3^{(i)}$$

and

$$x_{i+1} = x_i(1) = a_0^{(i)} + a_1^{(i)} + a_2^{(i)} + a_3^{(i)}$$

then,

$$\begin{cases} 2a_2^{(i)} + 3a_3^{(i)} = 3(x_{i+1} - x_{i+1}^{-}) - a_1^{(i)} \\ a_2^{(i)} + a_3^{(i)} = x_{i+1} - a_0^{(i)} - a_1^{(i)} \end{cases}$$

Also, from interpolation we know

$$\boxed{a_0^{(i)} = x_i(0) = x_i}$$
$$\boxed{b_0^{(i)} = y_i(0) = y_i}$$

Solving this system:

$$\begin{array}{cc} a_2^{(i)} & a_3^{(i)} \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & A \\ 2 & 3 & B \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & A \\ 0 & 1 & B-2A \end{array}\right]$$

$$a_3^{(i)} = B - 2A = 3\left(x_{i+1} - x_{i+1}^-\right) - a_1^{(i)}$$
$$- 2x_{i+1} + 2a_0^{(i)} + 2a_1^{(i)}$$
$$= \boxed{3x_{i+1}} - 3x_{i+1}^- + 3x_i^+ \boxed{-3x_i}$$
$$\boxed{-2x_{i+1}} + \boxed{2x_i}$$

or
$$\boxed{a_3^{(i)} = 3\left(x_i^+ - x_{i+1}^-\right) + x_{i+1} - x_i}$$

and $a_2^{(i)} = A - a_3^{(i)}$
$$= x_{i+1} - a_0^{(i)} - a_1^{(i)} - a_3^{(i)}$$
$$= \cancel{x_{i+1}} - \cancel{x_i} - 3(x_i^+ - x_i) - 3\left(x_i^+ - x_{i+1}^-\right) - \cancel{x_{i+1}} + \cancel{x_i}$$
$$= -6x_i^+ + 3x_i + 3x_{i+1}^- = 3\left(x_i + x_{i+1}^- - 2x_i^+\right)$$

or
$$\boxed{a_2^{(i)} = 3\left(x_i + x_{i+1}^- - 2x_i^+\right)}$$

Similarly,

$$b_3^{(i)} = 3\,(y_i^+ - y_{i+1}^-) + y_{i+1} - y_i$$

$$b_2^{(i)} = 3\,(y_i + y_{i+1}^- - 2\,y_i^+)$$

Summarizing given the data points:

Interpolation points: (x_i, y_i), (x_{i+1}, y_{i+1})

left guide point: (x_i^+, y_i^+)

Right guide point: (x_{i+1}^-, y_{i+1}^-)

we can construct cubic parametric polynomials:

$$\boxed{x_i(t) = a_0^{(i)} + a_1^{(i)} t + a_2^{(i)} t^2 + a_3^{(i)} t^3}$$
$$\boxed{y_i(t) = b_0^{(i)} + b_1^{(i)} t + b_2^{(i)} t^2 + b_3^{(i)} t^3}, \quad t \in [0,1].$$

Interpolating the points: (x_i, y_i) and (x_{i+1}, y_{i+1})

and satisfying the eight conditions:

$$x_i(0) = x_i \qquad y_i(0) = y_i$$
$$x_i(1) = x_{i+1} \qquad y_i(1) = y_{i+1}$$
$$x_i'(0) = 3(x_i^+ - x_i) \qquad y_i'(0) = 3(y_i^+ - y_i)$$
$$x_i'(1) = 3(x_{i+1} - x_{i+1}^-) \qquad y_i'(1) = 3(y_{i+1} - y_{i+1}^-)$$

They are given by

$$\boxed{\begin{aligned} x_i(t) = x_i + 3(x_i^+ - x_i)t + 3(x_i + x_{i+1}^- - 2x_i^+)t^2 \\ + \left(x_{i+1} - x_i + 3(x_i^+ - x_{i+1}^-)\right)t^3 \end{aligned}}$$

$$\boxed{\begin{aligned} y_i(t) = y_i + 3(y_i^+ - y_i)t + 3(y_i + y_{i+1}^- - 2y_i^+)t^2 \\ + \left(y_{i+1} - y_i + 3(y_i^+ - y_{i+1}^-)\right)t^3. \end{aligned}}$$

Alternative Derivation of parametric Bezier curves
DIVIDED DIFFERENCES USING HERMITE INTERPOLATION.

$$t_0 = 0 \qquad X_i[t_0] = X_i(0)$$

$$X_i[t_0,t_1] = X_i'(\overset{t_0}{0})$$

$$t_1 = 0 \qquad X_i[t_1] = X_i(0)$$

$$X_i[t_1,t_2] = \frac{X_i(1) - X_i(0)}{1}$$

$$t_2 = 1 \qquad X_i[t_2] = X_i(1)$$

$$X_i[t_2,t_3] = X_i'(1)$$

$$t_3 = 1 \qquad X_i[t_3] = X_i(1)$$

Next column:

$$X_i[t_0,t_1,t_2] = \frac{X_i[t_1,t_2] - X_i[t_0,t_1]}{t_2 - t_0}$$

$$= \frac{X_i(1) - X_i(0) - X_i'(0)}{1}$$

$$X_i[t_1,t_2,t_3] = \frac{X_i[t_2,t_3] - X_i[t_1,t_2]}{t_3 - t_1}$$

$$= \frac{X_i'(1) - X_i(1) + X_i(0)}{1}$$

$$X[t_0,t_1,t_2,t_3] = \frac{X_i[t_1,t_2,t_3] - X_i[t_0,t_1,t_2]}{t_3-t_1=1}$$

$$= \frac{X_i'(1) - X_i(1) + X_i(0) - X_i(1) + X_i(0) + X_i'(0)}{1}$$

$$= X_i'(1) - 2X_i(1) + 2X_i(0) + X_i'(0).$$

Now, using

$$X_i'(0) = 3(X_i^+ - X_i), \quad X_i'(1) = 3(X_{i+1} - X_{i+1}^-)$$

and interpolation points.

We obtain.

$$X_i(t) = X_i(0) + X_i'(0)t + (X_i(1) - X_i(0) - X_i'(0))t^2$$
$$+ (X_i'(1) - 2(\overset{=X_{i+1}}{X_i(1)} - \overset{=X_i}{X_i(0)}) + X_i'(0))t^2(t-1)$$

or

$$X_i(t) = X_i + 3(X_i^+ - X_i)t + (X_{i+1} - X_i - 3(X_i^+ - X_i))t^2$$
$$- (X_i'(1) - 2(X_{i+1} - X_i) + X_i'(0))t^2$$
$$+ (X_i'(1) - 2(X_{i+1} - X_i) + X_i'(0))t^3$$

$$=$$